TWISTED ALEXANDER POLYNOMIALS AND CHARACTER VARIETIES FOR 2-BRIDGE KNOTS (Twisted topological invariants and topology of low-dimensional manifolds)

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TWISTED ALEXANDER POLYNOMIALS AND CHARACTER VARIETIES FOR 2-BRIDGE KNOTS

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1. Introduction

In this extended abstract, we will survey the results in [8] by the authors. We note that this extended abstract contains no original results.

Since the twisted Alexander polynomial was introduced in the 90's [11, 14, 9], it has been successfully applied to many questions in knot theory and low dimensional topology. We refer the reader to the survey paper by Friedl and Vidussi [6] for more about the twisted Alexander polynomial.

One of the most remarkable applications of the twisted Alexander polynomials is to detect fiberedness of a knot: Friedl and Vidussi [5] showed that the twisted Alexander polynomials associated with finite representations detect fiberedness of a knot, and furthermore, fiberedness of 3-manifolds.

In [8] the authors considered another approach for detecting fiberedness of a knot: they used the $SL(2,\mathbb{C})$-character variety of a knot group and the twisted Alexander polynomial associated with it. We note that the idea of using the $SL(2,\mathbb{C})$-character variety for 3-manifold questions originates from Culler and Shallen [2]. For a knot, each coefficient of the twisted Alexander polynomial defines a complex valued function on the $SL(2,\mathbb{C})$-character variety of the knot group, and if the top coefficient function has value 1 for a character, then the character is called *monic*. It is known that every nonabelian $SL(2,\mathbb{C})$-character of a fibered knot is monic [7], and the main result in [8] is about the question asking if the converse holds. More precisely, in [8] the authors showed that for a nonfibered 2-bridge knot, there exists an irreducible curve component in the nonabelian $SL(2,\mathbb{C})$-character variety of the knot containing only a finite number of monic characters.

Although it is already known that the (classical) Alexander polynomial detects if a 2-bridge knot (and more generally an alternating knot) is fibered, the above result of the authors can be considered as a suggestion of a new approach for studying relationships between fiberedness of knots and twisted Alexander polynomials.

In Section 2, we review the character variety and the twisted Alexander polynomial of a 2-bridge knot, and we discuss the main results in [8] in Section 3.

2. Character varieties and twisted Alexander polynomials

2.1. **character variety of a 2-bridge knot.** Let $K=K(\alpha,\beta)$ be a 2-bridge knot where α and β are coprime integers with $-\alpha<\beta<\alpha$. Two 2-bridge knots $K(\alpha,\beta)$ and $K(\alpha',\beta')$ are isotopic if and only if $\alpha=\alpha'$ and $\beta\equiv\beta'$ or $\beta\beta'\equiv 1 \bmod \alpha$. It is well-known that the knot group $G(K)$ of K has a presentation

$$G(K)=\langle a,b\,|\,wa=bw\rangle,\quad w=a^{\epsilon_1}b^{\epsilon_2}\cdots a^{\epsilon_{\alpha-2}}b^{\epsilon_{\alpha-1}}$$

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where $\epsilon_i = (-1)^{[\frac{\beta}{\alpha}i]}$ and $[a]$ denotes the greatest integer less than or equal to $a \in \mathbb{R}$. Let a and b be the generators of G which represent the meridian up to conjugation. Let $\rho\colon G(K) \to SL(2.\mathbb{C})$ be a nonabelian representation of $G(K)$. Then for the matrices $C = \begin{pmatrix} s & 1 \\ 0 & s^{-1} \end{pmatrix}$ and $D = \begin{pmatrix} s & 0 \\ 2-y & s^{-1} \end{pmatrix}$, we may assume that $\rho(a) = C$ and $\rho(b) = D$ by taking conjugation if necessary. In fact, we have the following proposition due to Riley.

Proposition 2.1. [13, Theorem 1] *The assignment $\rho(a) = C$, $\rho(b) = D$ defines a nonabelian representation of $G(K)$ if and only if the pair (s, y) satisfies the equation*

$$w^{11} + (s^{-1} - s)w^{12} = 0,$$

where $W = \rho(w) = (w^{ij})$. Conversely, every nonabelian representation of $G(K)$ is conjugate to a representation satisfying the above equation.

Definition 2.2. The *Riley polynomial* of a 2-bridge knot K is the above polynomial $\phi(s,y) = w^{11} + (s^{-1} - s)w^{12} \in \mathbb{Z}[s^{\pm 1}, y]$.

For a finitely generated group G, We define $R(G) = \mathrm{Hom}(G, SL(2,\mathbb{C}))$. Then the $SL(2,\mathbb{C})$*-character variety* of G is defined to be the algebro-geometric quotient of $R(G)$ by the conjugate action, and we denote it by $X(G)$. For a representation $\rho \in R(G)$, the *character* of ρ is a map $\chi_\rho\colon G \to \mathbb{C}$ defined by $\chi_\rho(G) = \mathrm{tr}(\rho(\gamma))$ for $\gamma \in G$. Then it is known that there is a canonical identification $X(G) = \{\chi_\rho \mid \rho \in R(G)\}$.

Let $R^{\mathrm{nab}}(G)$ be the set of $\rho \in R(G)$ which is nonabelian. For the map $t\colon R(G) \to X(G)$ given by $t(\rho) = \chi_\rho$, we define $X^{\mathrm{nab}}(G)$ to be the image of $R^{\mathrm{nab}}(G)$ under t. For a knot K, we write $R(K)$ and $X(K)$ for $R(G(K))$ and $X(G(K))$, respectively, and similarly $R^{\mathrm{nab}}(K)$ and $X^{\mathrm{nab}}(K)$ for $R^{\mathrm{nab}}(G(K))$ and $X^{\mathrm{nab}}(G(K))$, respectively.

Let $K = K(\alpha, \beta)$ as above. For each $\gamma \in G(K)$, we define $t_\gamma\colon R(K) \to \mathbb{C}$ by $t_\gamma(\rho) = \mathrm{tr}(\rho(\gamma)))$. Then $X^{\mathrm{nab}}(K)$ is identified with the image of $R^{\mathrm{nab}}(K)$ under the map $(t_a, t_{ab^{-1}})\colon R(K) \to \mathbb{C}^2$ (see [2, Proposition 1.4.1] and [12, Senction 2]). Since $(t_a, t_{ab^{-1}}) = (s + s^{-1}, y)$, if ϕ is considered as a polynomial in $x = s + s^{-1}$ and y, then $X^{\mathrm{nab}}(K)$ is identified with $\{(x,y) \in \mathbb{C}^2 \mid \phi(x,y) = 0\}$.

2.2. **twisted Alexander polynomials.** For a knot group $G(K)$, we fix a Wirtinger persentation $G(K) = \langle \gamma_1, \ldots, \gamma_k \mid, r_1, \ldots, r_{k-1}\rangle$. Then following Wada [14], for a given representation $\rho\colon G(K) \to GL(2,\mathbb{C})$, one can define the *twisted Alexander polynomial* $\Delta_{K,\rho}(t) \in \mathbb{C}(t)$ which is well-defined up to multiplication by ϵt^{2i} ($\epsilon \in \mathbb{C}^*$, $i \in \mathbb{Z}$). In the case that ρ is a nonabelian special linear representation $\rho\colon G(K) \to SL(2,\mathbb{C})$, $\Delta_{K,\rho}(t) \in \mathbb{C}[t^{\pm 1}]$ [10, Theorem 3.1] and it is well-defined up to multiplication by t^{2i} ($i \in \mathbb{Z}$). We refer the reader to [14] for a precise definition of the twisted Alexander polynomial. We note that if ρ and η are conjugate $SL(2,\mathbb{C})$-representations, then $\Delta_{K,\rho}(t) = \Delta_{K,\eta}(t)$.

Since when we pick a nonabelian representation $\rho\colon G(K) \to SL(2,\mathbb{C})$ we obtain $\Delta_{K,\rho}(t)$ which is associated with ρ, each coefficient of $\Delta_{K,\rho}(t)$ can be considered as a $\mathbb{C}$-valued function on $R^{\mathrm{nab}}(K)$. Furthermore, each coefficient defines a $\mathbb{C}$-valued function on $X^{\mathrm{nab}}(K)$: if ρ and $\eta\colon G(K) \to SL(2,\mathbb{C})$ are nonabelian representations with $\chi_\rho = \chi_\eta$ such that ρ is irreducible, then ρ is conjugate to η (see [2, Proposition 1.5.2]), and hence $\Delta_{K,\rho}(t) = \Delta_{K,\eta}(t)$. And if ρ and η are reducible nonabelain, then they are determined by $\Delta_K(t)$ and hence $\Delta_{K,\rho}(t) = \Delta_{K,\eta}(t)$ (see the proof of [10, Theorem 3.1]). Therefore, we can define the *twisted Alexander polynomial associated with* $\chi \in X^{\mathrm{nab}}(K)$ to be $\Delta_{K,\rho}(t)$ where

$\chi = \chi_\rho$ and we denote it by $\Delta_{K,\chi}(t)$. We also say that a nonabelian representation $\rho\colon G(K) \to SL(2,\mathbb{C})$ (resp. a nonabelian character χ) is *monic* if $\Delta_{K,\rho}(t)$ (resp. $\Delta_{K,\chi}(t)$) is a monic polynomial. We note that for a 2-bridge knot K, each coefficient of $\Delta_{K,\rho}(t)$ and $\Delta_{K,\chi}(t)$ can be considered as a function of s and y or a function of x and y where $x = s + s^{-1}$.

3. Finiteness of monic characters

The following theorems are main results in [8]. We do not give the proofs of these theorems here and the reader is referred to [8] for the proofs. We also note that in [8] one can find more finiteness results and examples. The first theorem states that the twisted Alexander polynomials associated with all nonabelian $SL(2,\mathbb{C})$-representations detect fiberedness of a 2-bridge knot:

Theorem 3.1. [8, Theorem 4.1] *A 2-bridge knot K is fibered if and only if $\Delta_{K,\rho}(t)$ is monic for every nonabelian representation $\rho : G(K) \to SL(2,\mathbb{C})$.*

Basically the proof of Theorem 3.1 uses the existence of a reducible nonabelian representation of $G(K)$, which is due to Burde [1] and de Rham [3].

Since the (classical) Alexander polynomial detects fiberedness of a 2-bridge knot (and more generally an alternating knot), one might consider that Theorem 3.1 is not so helpful. But using Theorem 3.1 we obtain the following finiteness theorem, which seems more interesting.

Theorem 3.2. [8, Theorem 4.2] *For a nonfibered 2-bridge knot K, there exists an irreducible curve component in $X^{\mathrm{nab}}(K)$ which contains only a finite number of monic characters.*

As the (classical) Alexander polynomial gives the genus of a 2-bridge knot (and more generally an alternating knot), we also obtain the following finiteness theorem regarding the knot genus and twisted Alexander polynomials:

Theorem 3.3. [8, Theorem 4.3] *For a 2-bridge knot K of genus g, there exists an irreducible curve component X_1 in $X^{\mathrm{nab}}(K)$ such that $\deg(\Delta_{K,\chi}(t)) = 4g - 2$ for all but finitely many $\chi \in X_1$.*

Recently Dunfield, Friedl and Jackson [4] showed that for a hyperbolic knot K with at most 16 crossings and a lift ρ_0 of the discrete faithful representation $\rho\colon G(K) \to PSL(2,\mathbb{C})$ associated with K, the twisted Alexander polynomial $\Delta_{K,\rho_0}(t)$ detects fiberedness of the knot K. Moreover, it is known that the for a hyperbolic knot K, there is a *canonical component* $X_0(K)$ in $X(K)$ that is a curve containing ρ_0, and for any knot K, $X(K)$ contains a curve component. Therefore we suggest the following conjecture:

Conjecture 3.4. [8, Conjecture 6.4] For a nonfibered knot K, there exists a curve component $X_1(K)$ in $X^{\mathrm{nab}}(K)$ so that $\{\chi \in X_1(K) \mid \Delta_{K,\chi}(t) \text{ is monic}\}$ is a finite set.

References

[1] G. Burde, *Darstellungen von Knotengruppen*, Math. Ann. **173** (1967), 24–33.

[2] M. Culler and P. B. Shalen, *Varieties of group representations and splittings of 3-manifolds*, Ann. of Math. **117** (1983), 109–146.

[3] G. de Rham, *Introduction aux polynomes d'un noeud*, Enseignement Math. (2) **13** (1967), 187–194.

[4] N. Dunfield, S. Friedl and N. Jackson, *Twisted Alexander polynomials of hyperbolic knots*, in preparation.

[5] S. Friedl and S. Vidussi, *Twisted Alexander polynomials detect fibered 3-manifolds*, arXiv:0805.1234, to appear in Annals of Mathematics.

[6] S. Friedl and S. Vidussi, *A survey of twisted Alexander polynomials*, Proc. of the Conf., The Mathematics of Knots: Theory and Application, in Heidelberg 2008.

[7] H. Goda, T. Kitano and T. Morifuji, *Reidemeister torsion, twisted Alexander polynomial and fibered knots*, Comment. Math. Helv. **80** (2005), 51–61.

[8] T. Kim and T. Morifuji, *Twisted Alexander polynomials and character varieties of 2-bridge knot groups*, arXiv:1006.4285.

[9] P. Kirk and C. Livingston, *Twisted Alexander invariants, Reidemeister torsion, and Casson-Gordon invariants*, Topology **38** (1999), 635-661.

[10] T. Kitano and T. Morifuji, *Divisibility of twisted Alexander polynomials and fibered knots*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **4** (2005), 179–186.

[11] X. S. Lin, *Representations of knot groups and twisted Alexander polynomials*, Acta Math. Sin. (Engl. Ser.) **17** (2001), 361–380.

[12] M. L. Macasieb, K. L. Petersen and R. M. Van Luijk, *On character varieties of two-bridge knot groups*, arXiv:0902.2195, to appear in Proc. Lond. Math. Soc.

[13] R. Riley, *Nonabelian representations of 2-bridge knot groups*, Quart. J. Math. Oxford Ser. (2) **35** (1984), 191–208.

[14] M. Wada, *Twisted Alexander polynomial for finitely presentable groups*, Topology **33** (1994), 241–256.

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